Game Theory taking over the world of PDEs

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Outline

GAMES

Main research question

Question

What families of random games have a deterministic limit value?

Example: Continuous weigthed reachability

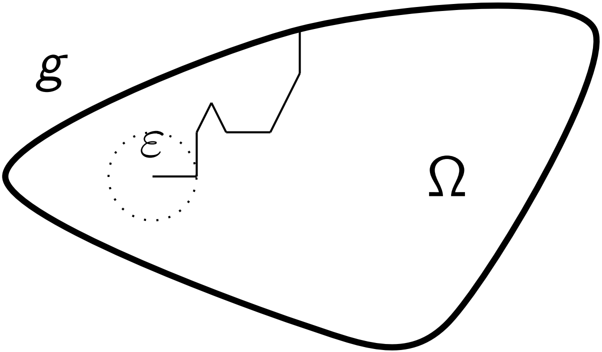

Figure 1: Continuous space reachability games

Dynamic

Consider the following dynamic, indexed by $\varepsilon > 0$

- State space is $\mathbb{R}^n$
- Domain $\Omega \subset \mathbb{R}^n$
- Reward function $g \colon \Omega^c \to \mathbb{R}$
- Initial position $x \in \Omega$
- Infinite random turn-based game
- At each turn, the corresponding player chooses where to move the state within $B(x, \varepsilon)$
- When arriving at $x \in \Omega^c$, min-player pays $g(x)$ to the max-player

Dynamic programming property

Let $u^\varepsilon \colon \Omega + B(0,\varepsilon) \to \mathbb{R}$ be the value. Then, for $x \in \Omega$,

$$u^\varepsilon(x) = \frac{1}{2}\left(\sup_{y \in B(x,\varepsilon)} u^\varepsilon(y), \inf_{y \in B(x,\varepsilon)} u^\varepsilon(y)\right) .$$

Asymptotic value

In this game,

$$(u^\varepsilon) \xrightarrow[\varepsilon \to 0]{} u\,,$$

such that u is the solution of

$$\begin{cases} \Delta_\infty u(x) = \sum_{i,j} \partial^2_{i,j} u(x) \partial_i u(x) \partial_j u(x) = 0 & x \in \Omega \\ u(x) = g(x) & x \in \partial\Omega \end{cases}$$

Research question

Question

What games in $\mathbb{R}^n$ have an asymptotic value?

The previous example shows that weigthed reachability has an asymptotic value.

Algorithmic perspective

For each $\varepsilon > 0$, this game is discrete in time and continuous in space.

Question

Is there a space discretization with probable approximation of the limit value function?

Answering this question requires proving rate of convergence in both continuous and discrete time settings.

PDEs

Example: Heat equation

Consider the heat equation.

$$\begin{cases} \partial_t u(t,x) - \Delta u(t,x) = \partial_t u(t,x) - \sum_i \partial^2_{i,i} u(t,x) = 0 & x \in \Omega \\ u(0,x) = u_0(x) & x \in \Omega \end{cases}$$

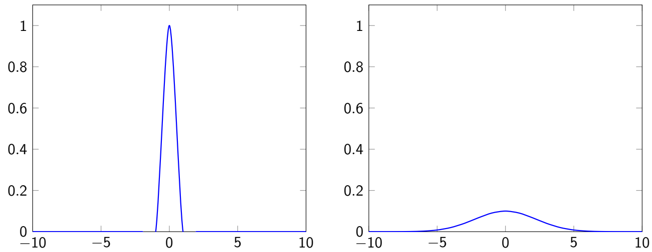

Figure 2: Heat kernel evolution

Heat equation in heterogeneous media

Consider the heat equation in a media.

$$\begin{cases} \partial_t u - \nabla(A \nabla u) = \partial_t u - \sum_i \partial_i \sum_j A_{i,j} \partial_j u = 0 & x \in \Omega \\ u(0,x) = u_0(x) & x \in \Omega \end{cases}$$

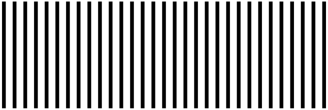

Figure 3: Heterogeneous periodic media

Heat equation: Homogenization limit

Consider the heat equation in a limit media. Let $\varepsilon > 0$,

$$\begin{cases} \partial_t u^\varepsilon - \nabla(A(x/\varepsilon)\nabla u^\varepsilon) = \partial_t u - \sum_i \partial_i \sum_j A_{i,j}(x/\varepsilon)\partial_j u^\varepsilon = 0 & x \in \Omega \\ u^\varepsilon(0,x) = u_0(x) & x \in \Omega \end{cases}$$

And consider the limit of u_ε, as $\varepsilon \to 0$.

Homogenization definition

Let $H \colon \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$ be a *"hamiltonian"*. Define, for $\varepsilon > 0$,

$$\begin{cases} \partial_t u^\varepsilon + H(\nabla u^\varepsilon, x/\varepsilon) = 0 & x \in \Omega \\ u^\varepsilon(0,x) = u_0(x) & x \in \Omega \end{cases}$$

Definition (Homogenization)

The hamiltonian H presents *homogenization* if there is an *effective* hamiltonian $\overline{H}$ such that $(u^\varepsilon) \xrightarrow[\varepsilon \to 0]{} u$, where u is the solution of

$$\begin{cases} \partial_t u + \overline{H}(\nabla u) = 0 & x \in \Omega \\ u(0,x) = u_0(x) & x \in \Omega \end{cases}$$

Homogenization question

Question

What Hamiltonians do homogenize?

Theorem (Sufficient conditions)

If H is periodic in the space variable, H homogenizes.

Stochastic homogenization

Since random medias are "periodic", consider now a random hamiltonian.

Definition (Stochastic Homogenization)

The random hamiltonian H presents homogenization if there is an *effective* hamiltonian $\overline{H}$ such that $(U^\varepsilon) \xrightarrow[\varepsilon \to 0]{} u$, where u is the (deterministic) solution of

$$\begin{cases} \partial_t u + \overline{H}(\nabla u) = 0 & x \in \Omega \\ u(0,x) = u_0(x) & x \in \Omega \end{cases}$$

Stochastic Homogenization question

Question

What random Hamiltonians present stochastic homogenization?

Theorem (Sufficient conditions)

Under standard assumptions on H and ***convexity*** *on the space variable, H has stochastic homogenization.*

Theorem (Tightness on convexity)

There exists H astisfying standard assumptions but not convexity on the space variable, such that H does not have stochastic homogenization.

GAMES and PDEs

Game values and Hamiltonians

In (random) games, we ask if

$$(V^\varepsilon) \xrightarrow[\varepsilon \to 0]{} v\,.$$

In stochastic homogenization, we ask if

$$(H(\nabla u^\varepsilon, x/\varepsilon)) \xrightarrow[\varepsilon \to 0]{} \overline{H}(\nabla u)\,.$$

Given a hamiltonian H, one may construct a game where the solution u^ε is the value of the game.

Question

Which hamiltonians have a game-theoretical interpretation?

Main research question

Question

What families of random games have a deterministic limit value?

GAMES in the plane

Random game on the plane

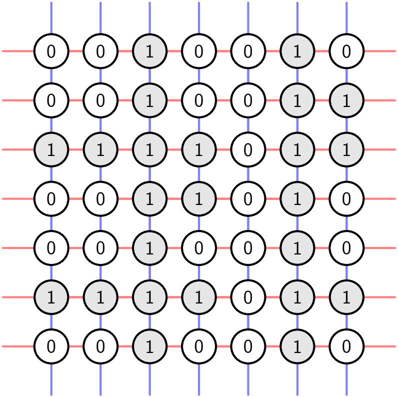

Figure 4: Average payoff game in random media

Dynamic

Consider the following dynamic, indexed by $\varepsilon > 0$

- State space is $\mathbb{Z}^2$
- Random reward function $G \colon \mathbb{Z} \to \mathbb{R}$, where $G(z) \sim B(p)$, for $p \in [0, 1]$
- Initial state is $(0, 0)$
- Infinite turn-based game
- At each turn, the corresponding player chooses where to move the state:
  - Max-player chooses *up* or *down*
  - Min-player chooses *left* or *right*
- The reward is the discounted average

$$\varepsilon \sum_{n \in \mathbb{N}} (1 - \varepsilon)^n G(z_n) \,.$$

Oriented version

If you restrict the player to move in one direction, i.e. a state is never visited again, then [GZ21] show that (V^ε) converges to a deterministic limit.

The general framework is still open.

Question

Is there a limit value? Is the limit a constant?

Percolation thresholds

Theorem

There exists $0 < p_0 < p_1 < 1$ *such that*

$$(V^\varepsilon) \xrightarrow[\varepsilon \to 0]{} 0 \qquad \forall p < p_0$$

$$(V^\varepsilon) \xrightarrow[\varepsilon \to 0]{} 1 \qquad \forall p > p_1$$

References I

Guillaume Garnier and Bruno Ziliotto.
Percolation games.
2021.